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Fish migration as a stochastic optimal stopping problem: application of the methodology in mathematical finance

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1. Introduction

Fish migration is a key biological phenomenon that drives food-webs. This is a large-scale mass biological phenomenon by coordinated behavior of fish population depending on environmental cues (Cote et al., 2017). A life history of migratory fish involves long-distant migrations between habitats, which are driven by many ecological factors, such as seasonal changes of habitat quality, existence of predators, and possibly by environmental changes due to climate changes and human activities (Chapman et al., 2015; Hagelin et al., 2016; Hulthén et al., 2015; Jones and Petreman, 2015). Comprehension and assessment of fish migration is necessary for better, ecologically-friendly water environment. In addition, prediction of timing and amount of fish migration is indispensable for sustainable fisheries management (Kumar and Kumari, 2015; Willis et al., 2015).

Population dynamics of animals like fish is an inherently stochastic biological phenomenon due to the lack of data and unresolved, possibly nonlinear, complicated, and multi-scale environmental and ecological processes (Buiatti et al., 2013; Lande et al., 2003; Lv and Pitchford, 2007). There exist a variety of costs for migration between habitats, such as physiological energy consumption (Yoshioka, 2016; Yoshioka, 2017) and predation (Brönmark et al., 2013; Chapman et al., 2013). Existence of physical barriers, such as weirs and dams equipped with poorly designed fishways would be obstacles of migration (Dugan et al., 2010; Leeuwen et al., 2016; Franklin and Hodges, 2015; Yoshioka et al., 2016). Fish migration between habitats is thus considered as a stochastic optimal stopping problem where a stopping time to be optimized corresponds to timing of migration. Ai and Sun (2012) discussed solvability of optimal stopping problem of a population governed by a stochastic logistic-like SDE. Yakubu and Fogarty (2006) formulated fish migration in terms of a spatially-discrete meta-population dynamics model with directional movements.

As an application example of a mathematical finance methodology to fish migration, this paper introduces a tractable stochastic optimal stopping problem. The decision-maker of the problem is a fish population. The present model assumes a non-structured population in the sense that the population contains the individuals of the same age. This assumption is not so restrictive for migration of major migratory fishes having a life history of single year such as *Plecoglossus altivelis* (*P. altivelis*, Ayu) serving as important inland fishery resources in Japan. Even for migratory fishes with a structured population such as salmonids, the model can be applied to analysis of their migration assuming that the population has a dominant age. Environmental cues that trigger the migration are implicitly incorporated into the coefficients of the present model, and their explicit and detailed mathematical modeling is beyond the scope of this paper. The exact solution to the present system of variational inequalities (VIs) are analytically expressed. Ecological characterization of the sufficient condition to guarantee the existence of the exact solution is presented, which provides a useful connection between mathematics and fish migration. Similarities between the present ecological problem and a problem of mathematical finance are discussed as well.

The rest of this paper is organized as follows. Section 2 presents the mathematical model focused in this paper. Section 3 gives a heuristic exact solution to the VI associated with the present model, proves its viscosity property, and shows that the solution is the unique viscosity solution to the VI. Section 4 poses advanced topics on mathematical modeling of fish migration and discusses relationship between the topics and tools in mathematical finance. Section 5 concludes this paper.

2. Mathematical model

2.1 Stochastic differential equation

A system of stochastic differential equations (SDEs) that governs the population dynamics, which is represented by the total number of population $N_t\left(\geq 0\right)$ in the habitat and their representative body weight $X_t\left(\geq 0\right)$ at the time t. Assume that there is no reproduction and the population is not structured. The governing equations of N_t and X_t are set as the Ito’s SDEs

$$\mathrm{d}N_t = -RN_t\mathrm{d}t\,,\quad t>0\,,\quad N_0 = n\,, \tag{1}$$

$$\mathrm{d}X_t = X_t\left(r\mathrm{d}t + \sigma\mathrm{d}B_t\right),\quad t>0\,,\quad X_0 = x \tag{2}$$

where B_t is the 1-D standard Brownian motion, $R>0$ is the natural mortality rate, $\sigma\geq 0$ is the environmental noise intensity, and $r>0$ is the intrinsic growth rate. σ is a lumped parameter that represents internal and external stochasticity involved in the population dynamics that affects the growth of individuals, such as competitions among individuals and fluctuations of environmental conditions of the habitat. Owing to the linearity of the SDEs, the conventional Ito's lemma leads to the SDE of the biomass $Z_t = N_t X_t$ as

$$\mathrm{d}Z_t = Z_t\left(\left(r-R\right)\mathrm{d}t + \sigma \mathrm{d}B_t\right),\quad t>0,\quad Z_0 = nx. \tag{3}$$

We assume the condition

$$r - R > \frac{\sigma^2}{2}, \tag{4}$$

so that the extinction of the population does not occur, namely $Z_t > 0$ for $t > 0$ almost surely when $Z_0 > 0$. Hereafter, the notation $z = nx$ is employed for the sake of brevity of descriptions.

2.2 Performance index

The performance index J to be maximized by the population, the decision-maker, is set as

$$J(z;\tau) = \mathrm{E}^z\left[\int_0^\tau \frac{q}{1-\alpha} Z_s^{1-\alpha} e^{-\delta s}\mathrm{d}s + \chi_{\{\tau<+\infty\}} e^{-\delta\tau}\left(\frac{1}{1-\alpha}Z_\tau^{1-\alpha} - k\right)\right] \tag{5}$$

where $\delta > 0$ is the discount rate, $q > 0$ is the weight parameter, $0 < \alpha < 1$ is the constant of sensitivity, $k > 0$ is the cost of migration, and $\chi_{\{\tau<+\infty\}}$ is the indicator function for the set $\tau \in [0,+\infty)$. The first term of J represents the cumulated benefit during the growth in the current habitat. The second term represents the benefit by reaching the next habitat through a migration and the cost of migration.

2.3 Value function

The value function $\Phi = \Phi(n,x)$ is the maximized performance index defined as

$$\Phi(z) = \sup_{\tau\in\Gamma} J(z;\tau) \tag{6}$$

where Γ is a set of non-negative, adapted stopping times. The main assumption made throughout this paper is as follows

Assumption 2.1

$$\lambda = \delta - (1-\alpha)(r-R) + \alpha(1-\alpha)\frac{\sigma^2}{2} > 0. \tag{7}$$

Assumption 2.1 is satisfied if δ is sufficiently large. This assumption is necessary to have a non-trivial optimal strategy. In fact, without this (namely when $-\lambda \geq 0$), we have an unbounded value function with $\tau^* = +\infty$ (no migration):

$$
\begin{aligned}
\Phi(z) &\geq J(z;+\infty) \\
&= \mathrm{E}^z\left[\int_0^{+\infty} \frac{q}{1-\alpha} Z_s^{1-\alpha} e^{-\delta s} \mathrm{d}s\right] \\
&= \frac{q}{1-\alpha} z^{1-\alpha} \int_0^{+\infty} \mathrm{E}^z\left[e^{(1-\alpha)(r-R)s+(1-\alpha)\sigma B_s}\right] e^{-\delta s} \mathrm{d}s \,. \\
&= \frac{q}{1-\alpha} z^{1-\alpha} \int_0^{+\infty} e^{-\lambda s} \mathrm{d}s \\
&= +\infty
\end{aligned}
\tag{8}
$$

The upper- and lower-bounds of the value function Φ are obtained as follows.

Proposition 2.1

$$
0 \leq \Phi(z) \leq \left(A + \frac{1}{1-\alpha}\right) z^{1-\alpha}, \quad z \geq 0, \quad A = \frac{q}{(1-\alpha)\lambda} > 0 \,. \tag{9}
$$

(Proof of Proposition 2.1)

The lower-bound that shows non-negativity of Φ is trivial by the functional form of J. On the other hand, the upper-bound is obtained as follows.

$$
\begin{aligned}
\Phi(z) &\leq \mathrm{E}^z\left[\int_0^{+\infty} \frac{q}{1-\alpha} Z_s^{1-\alpha} e^{-\delta s} \mathrm{d}s\right] + \sup_{\tau\in\Gamma} \mathrm{E}^z\left[\chi_{\{\tau<+\infty\}} e^{-\delta\tau}\left(\frac{1}{1-\alpha} Z_\tau^{1-\alpha} - k\right)\right] \\
&= Az^{1-\alpha} + \sup_{\tau\in\Gamma} \mathrm{E}^z\left[e^{-\delta\tau}\left(\frac{1}{1-\alpha} Z_\tau^{1-\alpha} - k\right)\right] \\
&\leq Az^{1-\alpha} + \frac{1}{1-\alpha} \sup_{t\geq 0} \mathrm{E}^z\left[Z_t^{1-\alpha} e^{-\delta t}\right]
\end{aligned}
\quad . \tag{10}
$$

Since

$$
\sup_{t\geq 0} \mathrm{E}^z\left[Z_t^{1-\alpha} e^{-\delta t}\right] = z^{1-\alpha} \sup_{t\geq 0}\left\{e^{-\lambda t}\right\} = z^{1-\alpha}, \tag{11}
$$

combining (10) and (11) yields the desired upper-bound, and thus the proof is completed.

□

Corollary 2.1

*By **Proposition 2.1**, Φ is continuous at the origin $z = 0$.*

The value function Φ is continuous with respect to z, which is proven by the following non-standard continuity result.

Proposition 2.2

$$\left|\Phi(z_1)-\Phi(z_2)\right| \le \left(A+\frac{1}{1-\alpha}\right)\left|z_1^{1-\alpha}-z_2^{1-\alpha}\right|,\quad z_1,z_2\ge 0. \tag{12}$$

(Proof of Proposition 2.2)

By the definition of Φ, we have

$$\begin{aligned}\left|\Phi(z_1)-\Phi(z_2)\right| &= \left|\sup_{\tau\in\Gamma} J(z_1;\tau)-\sup_{\tau\in\Gamma} J(z_2;\tau)\right| \\ &\le \left|\sup_{\tau\in\Gamma}\left(J(z_1;\tau)-J(z_2;\tau)\right)\right| \\ &\le \sup_{\tau\in\Gamma}\left|J(z_1;\tau)-J(z_2;\tau)\right|\end{aligned} \tag{13}$$

and

$$\begin{aligned}&J(z_1;\tau)-J(z_2;\tau)\\ &=\mathrm{E}^{z_1}\left[\int_0^\tau \frac{q}{1-\alpha}Z_s^{1-\alpha}e^{-\delta s}\mathrm{d}s+\chi_{\{\tau<+\infty\}}e^{-\delta\tau}\left(\frac{1}{1-\alpha}Z_\tau^{1-\alpha}-k\right)\right]\\ &-\mathrm{E}^{z_2}\left[\int_0^\tau \frac{q}{1-\alpha}Z_s^{1-\alpha}e^{-\delta s}\mathrm{d}s+\chi_{\{\tau<+\infty\}}e^{-\delta\tau}\left(\frac{1}{1-\alpha}Z_\tau^{1-\alpha}-k\right)\right].\\ &\le\frac{q}{1-\alpha}\left|z_1^{1-\alpha}-z_2^{1-\alpha}\right|\mathrm{E}\left[\int_0^\tau e^{(1-\alpha)(r-R)s+(1-\alpha)\sigma B_s}e^{-\delta s}\mathrm{d}s\right]\\ &+\frac{1}{1-\alpha}\left|z_1^{1-\alpha}-z_2^{1-\alpha}\right|\mathrm{E}\left[e^{-\lambda\tau}\right]\end{aligned} \tag{14}$$

We then have

$$\begin{aligned}\left|\Phi(z_1)-\Phi(z_2)\right| &\le \frac{1}{1-\alpha}\left|z_1^{1-\alpha}-z_2^{1-\alpha}\right|\sup_{\tau\in\Gamma}\left|\begin{matrix}\mathrm{E}\left[\int_0^\tau e^{(1-\alpha)(r-R)s+(1-\alpha)\sigma B_s}e^{-\delta s}\mathrm{d}s\right]\\ +\mathrm{E}\left[e^{-\lambda\tau}\right]\end{matrix}\right| \\ &\le \frac{1}{1-\alpha}\left|z_1^{1-\alpha}-z_2^{1-\alpha}\right|\left|\begin{matrix}\sup_{\tau\in\Gamma}\mathrm{E}\left[\int_0^\tau e^{(1-\alpha)(r-R)s+(1-\alpha)\sigma B_s}e^{-\delta s}\mathrm{d}s\right]\\ +\sup_{\tau\in\Gamma}\mathrm{E}\left[e^{-\lambda\tau}\right]\end{matrix}\right|, \\ &= \left(A+\frac{1}{1-\alpha}\right)\left|z_1^{1-\alpha}-z_2^{1-\alpha}\right|\end{aligned} \tag{15}$$

which is the desired estimate (12).

□

Combining **Propositions 2.1** and **2.2** immediately shows the following theorem, which is necessary for a verification of the VI.

Theorem 2.1

The value function $\Phi = \Phi(z)$ *is continuous and locally bounded for* $z \geq 0$.

2.4 Variational inequality

Application of the dynamic programming principle to (6) leads to the VI

$$\min\left\{L\Phi - \frac{q}{1-\alpha}z^{1-\alpha}, \Phi - \left(\frac{1}{1-\alpha}z^{1-\alpha} - k\right)\right\} = 0, \quad z > 0 \tag{16}$$

with the degenerate elliptic operator L given by

$$L = \delta - (r - R)z\frac{\mathrm{d}}{\mathrm{d}z} - \frac{1}{2}\sigma^2 z^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2}, \tag{17}$$

subject to the boundary condition $\Phi(0) = 0$, meaning a trivial fact that there is no profit when no population exists in the current habitat.

2.5 Financial interpretation

The present mathematical model has a financial interpretation. The stochastic process Z_t is interpreted as a value process of some project. The first term of the performance index J represents the cumulative profit of the project, and the second term is a sum of the terminal profit and the exit cost. Owing to this mathematical similarity between the ecological and financial problems, mathematical tools developed in mathematical finance can be effectively applied to the present problem as demonstrated in the next section.

It should be noted that one of the significant differences between the present ecological model and the conventional financial models is on the decision-makers. In the present model, the decision-maker is the population, which is the controlled stochastic process itself, while it is not the case for the financial models where the decision-maker is typically an observer of the process to be controlled.

3. Exact solution

3.1 A candidate of exact solutions

We have a heuristic, □almost□classical solution $\Phi_0 = \Phi_0(z)$ ($z \geq 0$) to the VI(16).

Proposition 3.1

Assume $\lambda > q$. *Then, the function* $\Phi_0(z)$ *defined below satisfies the VI(16) in the classical sense expect at the one point* $z = \bar{z}$. *In addition, this* Φ_0 *complies with the boundary condition* $\Phi_0(0) = 0$*:*

$$\Phi_0(z)=\begin{cases} Az^{1-\alpha}+Bz^{\omega} & (0\le z\le \bar{z}) \\ \dfrac{1}{1-\alpha}z^{1-\alpha}-k & (z>\bar{z}) \end{cases}, \tag{18}$$

$$\bar{z}=\left[\frac{\lambda\omega k}{\lambda-q}\cdot\frac{1-\alpha}{\omega-1+\alpha}\right]^{\frac{1}{1-\alpha}}>0, \tag{19}$$

$$\omega=\frac{1}{\sigma^2}\left[-\left(r-R-\frac{\sigma^2}{2}\right)+\sqrt{\left(r-R-\frac{\sigma^2}{2}\right)^2+2\sigma^2\delta}\right]>1-\alpha, \tag{20}$$

$$B=\frac{k(1-\alpha)}{\omega-1+\alpha}\left(\frac{1}{\bar{z}}\right)^{\omega}>0, \tag{21}$$

and A *given in (9).*

The regularity results $\Phi_0\in C[0,+\infty)\cap C^1(0,+\infty)$ and $\Phi_0\in C^2\left((0,\bar{z})\cup(\bar{z},+\infty)\right)$ hold true by **Proposition 3.1**. In addition, Φ_0 complies with **Propositions 2.1** and **2.2**. Since $\Phi_0\notin C[0,+\infty)\cap C^2(0,+\infty)$, it is not a classical solution to the VI(16). However, it turns out that the function Φ_0 is a continuous viscosity solution to the VI(16).

If the function Φ_0 is the value function Φ, based on the knowledge of mathematical finance, its implication is that it is optimal to migrate from the current to the next habitat at $\tau=\tau^*$ such that

$$\tau^*=\inf\left\{t>0 \mid Z_t\ge\bar{z}, Z_0=z\right\}. \tag{22}$$

For small $z<\bar{z}$, the result implies

$$\tau^*=\inf\left\{t>0 \mid Z_t=\bar{z}, Z_0=z\right\}. \tag{23}$$

A remark on the assumption $\lambda>q$ is provided here. If this assumption does not hold, then there is no solution to the VI(16) of the form (18). A solution with $\lambda\le q$ is formally obtained as

$$\Phi_0(z)=Az^{1-\alpha} \tag{24}$$

since

$$\lim_{q\to\lambda-0}\bar{z}=+\infty \text{ and } \lim_{q\to\lambda-0}B=0. \tag{25}$$

The function Φ_0 in (24) implies $\tau^*=+\infty$, namely, it is optimal to stay in the current habitat. Note that the function Φ_0 in (24) is a classical solution to the VI(16) that belongs to

$C[0,+\infty)\cap C^2(0,+\infty)$.

3.2 Viscosity property

A definition of viscosity solutions to the VI(16) is presented, which is utilized to verify the viscosity property of Φ_0.

Definition 3.1

A continuous function $\Phi=\Phi(z)$ *for* $z\geq 0$ *such that* $\Phi(0)=0$ *is a viscosity sub-solution (super-solution) to the VI(16) if for each* $z=\hat{z}>0$, *the inequality*

$$\min\left\{Lw-\frac{q}{1-\alpha}z^{1-\alpha},w-\left(\frac{1}{1-\alpha}z^{1-\alpha}-k\right)\right\}\leq 0\ (\geq 0)\quad at\quad z=\hat{z} \tag{26}$$

holds true for any test function $w\in C^2(0,+\infty)$ *such that* $w\geq\Phi$ *(*$w\leq\Phi$*) and* $w-\Phi$ *attains a local minimum (maximum) at* $z=\hat{z}$. *A continuous function* $\Phi=\Phi(z)$ *for* $z\geq 0$ *such that* $\Phi(0)=0$ *is a viscosity solution if it is a viscosity sub-solution as well as a viscosity super-solution.*

□

Proposition 3.2

Assume $\lambda>q$. Φ_0 *in **Proposition 3.1** is a viscosity solution to the VI(16).*

(Proof of Proposition 3.2)

It is sufficient to check the viscosity property of Φ_0 only at $z=\bar{z}$. To see that Φ_0 is a viscosity sub-solution is trivial since the condition of viscosity sub-solutions reduces to

$$\min\left\{Lw-\frac{q}{1-\alpha}z^{1-\alpha},0\right\}\leq 0\quad\text{at}\quad z=\bar{z} \tag{27}$$

for any test functions w for viscosity sub-solutions. The left-hand side of (27) is non-positive for any w.

To see Φ_0 satisfies the condition of viscosity super-solution at $z=\bar{z}$, it is sufficient to check

$$\min\left\{Lw-\frac{q}{1-\alpha}z^{1-\alpha},0\right\}\geq 0\quad\text{at}\quad z=\bar{z}, \tag{28}$$

namely

$$Lw-\frac{q}{1-\alpha}z^{1-\alpha}\geq 0 \text{ at } z=\bar{z} \tag{29}$$

for any test functions w for viscosity super-solutions. Since Φ_0 is continuously differentiable at $z=\bar{z}$, w should satisfy $w(\bar{z})=\Phi_0(\bar{z})$ and $\frac{\mathrm{d}w}{\mathrm{d}z}(\bar{z})=\frac{\mathrm{d}\Phi_0}{\mathrm{d}z}(\bar{z})$. In addition, we have

$$\delta\Phi_0(\bar{z})-(r-R)z\frac{\mathrm{d}\Phi_0}{\mathrm{d}z}(\bar{z})-\frac{1}{2}\sigma^2 z^2\frac{\mathrm{d}^2\Phi_0}{\mathrm{d}z^2}(\bar{z}-0)-\frac{q}{1-\alpha}\bar{z}^{1-\alpha}=0. \tag{30}$$

By $\bar{z}>0$, combining (29) and (30) shows that it is sufficient to show

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}(\bar{z})\leq\frac{\mathrm{d}^2\Phi_0}{\mathrm{d}z^2}(\bar{z}-0) \tag{31}$$

against any test functions w for viscosity super-solutions. Such a w has to satisfy

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}(\bar{z})\leq\min\left\{\frac{\mathrm{d}^2\Phi_0}{\mathrm{d}z^2}(\bar{z}-0),\frac{\mathrm{d}^2\Phi_0}{\mathrm{d}z^2}(\bar{z}+0)\right\} \tag{32}$$

by **Definition 3.1** and the fact that $w-\Phi_0$ attains a local maximum at $z=\bar{z}$. Therefore, the condition (31) is satisfied by the test function that complies with (32). The result implies that Φ_0 is a viscosity solution to the VI(16).

□

Similarly, we also have the following result.

Proposition 3.3

Assume $\lambda\leq q$. Then, Φ_0 in (24) is a viscosity solution to the VI(16).

3.3 Verification

An application of Theorem 2.1 of Reikvam (1998) with slight modifications show the following theorem. An idea of its proof is also presented.

Theorem 3.1

Φ_0 is the value function Φ.

(Idea of the Proof of Theorem 3.1)

The result of Theorem 2.1 of Reikvam (1998) holds true with the following modifications.

- ✓ Replace X_t and x in the literature by Z_t and z.
- ✓ Replace $L=(r-R)z\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{2}\sigma^2 z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}$ in the literature by

$$L=\delta+(r-R)z\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{2}\sigma^2 z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}.$$

- ✓ Replace $f(X_t)$ by $f(X_t)e^{-\delta t}$.
- ✓ Replace $g(X_\tau)$ by $g(X_\tau)e^{-\delta\tau}$.

□

Consequently, it is shown that Φ_0, which is an explicit viscosity solution, is the value function Φ. A remaining question is that whether Φ_0 is a unique viscosity solution to the VI(16) or not.

3.4 Uniqueness

As in the verification result, an application of Theorem 3.1 of Reikvam (1998) with slight modifications show the following theorem since $\tau^* < +\infty$ and $\{\Phi(Z_\tau)\}_{\tau\in\Gamma}$ is uniformly integrable for all $z \geq 0$.

Theorem 3.1

$\Phi = \Phi_0$ *is the unique viscosity solution to the VI(16).*

4. Advanced topics

4.1 Lévy noise

The equation (1) is an ordinary differential equation, which can be naturally extended as

$$\mathrm{d}N_t = -N_{t-0}\left(R\mathrm{d}t + \mathrm{d}V_t\right),\ t>0,\ N_0 = n \tag{33}$$

where V_t is a subordinator such as a compound Poisson process with positive jumps. This type of geometric Lévy processes have been applied to economic modeling related to portfolio optimization problems (Aït-Sahalia et al., 2009; Buckley et al., 2016; Pasin and Vargiolu, 2010;). In this case, the second term represents the discontinuous decrease of the population such as due to predation by waterfowls. If (1) is replaced by (33), a non-local term is added to the VI(16). Assuming $\delta > 0$ is sufficiently large, a candidate of almost classical exact solutions to the integro-differential VI is obtained as in **Proposition 3.1** where the degrees and the coefficients of the solutions change, but their qualitative structure remains the same. In addition, the viscosity property of the candidate is also explicitly verified, and it turns out to be the value function by Theorem 2.2 of Øksendal and Sulem (2005). Therefore, incorporating a Lévy process, into the present model in the above-mentioned manner does not encounter significant mathematical difficulties.

4.2 Information delay

Population dynamics subject to delayed information can be a reasonable approach for analyzing fish migration because theoretical analysis results based on SDEs implies that environmental changes possibly cause time delays in population dynamics (Solbu et al., 2013). Linkages such as transformation formulas between problems with and without refractions have been studied in Øksendal (2005), which provide key mathematical techniques to construct a solution to the present optimal stopping problem. In the framework of the present mathematical model, the delay can be incorporated into the definition of the value function Φ as

$$\Phi(z) = \sup_{\tau \in \Gamma_\theta} J(z;\tau) \tag{34}$$

where Γ_θ is the set of stopping times τ adapted to the filtration generated by the Brownian motion B_t such that $\tau \geq \theta > 0$. A new parameter θ appears in the extended model and the problem results in more complicated; however, Theorem 2.1 of Øksendal (2005) shows that the value function Φ in (34) can be rewritten as a problem without apparent delay. In fact, application of Theorem 2.1 of Øksendal (2005) to (34) shows

$$\Phi(z) = \sup_{\tau \in \Gamma} \mathrm{E}^z \left[\int_0^\tau \frac{q}{1-\alpha} Z_s^{1-\alpha} e^{-\delta s} \mathrm{d}s + e^{-\delta \tau} \chi_{\{\tau < +\infty\}} \Psi(Z_\tau) \right] \tag{35}$$

with

$$\Psi(z) = \mathrm{E}^z \left[\int_0^\theta \frac{q}{1-\alpha} Z_s^{1-\alpha} e^{-\delta s} \mathrm{d}s + e^{-\delta\theta} \left(\frac{1}{1-\alpha} Z_\theta^{1-\alpha} - k \right) \right]. \tag{36}$$

The right-hand side of (36) can be explicitly expressed as a polynomial of z since Z_t is a geometric Brownian motion. In fact, we have

$$\Psi(z) = \frac{q + (\lambda - q) e^{-\lambda\theta}}{(1-\alpha)\lambda} z^{1-\alpha} - k e^{-\delta\theta}. \tag{37}$$

Therefore, $\Psi(Z_\tau)$ in (35) is a polynomial of Z_τ. This implies that the boundedness and continuity results like **Propositions 2.1** and **2.2** hold true for the problem with the information delay. In addition, the free boundary $z = \bar{z}$ would not be found analytically.

4.3 Multiple optimal stopping

A life history of a fish typically contains many migrations. For example, *P. altivelis* in Japan has the spring-juvenile-upstream migration from sea to river midstream for growth, and the autumn downstream migration from river-midstream to river-downstream for spawning. The presented mathematical model describes one migration event, and it cannot be directly applied to the problem with multiple migration events. The concept of multiple optimal stopping, which has been investigated in mathematical finance and related research fields (Aleksandrov and Hambly, 2010; Carmona and Touzi, 2008; Christensen and Lempa, 2015; Leung et al., 2015; Yamazaki, 2015) can be employed to tackle this issue. In this framework, a migration strategy is characterized with sequential stopping times, and derivation of an optimal migration strategy reduces to solving a cascading system of VIs. We have found that it is possible to formulate a tractable multiple optimal stopping problem for fish migration where the solution to the system of VIs are expressed explicitly with coefficients uniquely determined from (uniquely solvable) nonlinear algebraic equations.

4.4 Ambiguity

Fish migration may be a decision-making problem of a population subject to model ambiguity, in which the population make decisions based on a biased model. The concept of multiplier robust control (Hansen and Sargent, 2006) has been an effective mathematical tool for modeling decision-making under ambiguity (Jang et al., 2016; Tsujimura, 2016; Zhang et al., 2017). Yoshioka and Yaegashi (2017b) have recently approached this issue numerically.

4.5 Numerical approximation

The present model can be made more realistic, but the resulting model will not be exactly solvable. For example, the SDE(2) can be replaced by the logistic counterpart

$$\mathrm{d}X_t = X_t\left(r\left(1 - X_t\right)\mathrm{d}t + \sigma \mathrm{d}B_t\right),\ \ t > 0,\ \ X_0 = x \tag{38}$$

with the upper-bounded deterministic growth rate such that $r\left(1 - X_t\right) \le r$ for $X_t \ge 0$. In this case, the upper- and lower-bounds of the value function Φ are obtained explicitly, but the associated VI turns out to be not exactly solvable. In such a case, a numerical scheme with stability and consistency properties (Forsyth and Labahn, 2007) can be used for solving the VI. A practical problem is to construct a computationally efficient numerical method for high-dimensional VIs like that in Darbon and Osher (2016).

5. Conclusions

An exactly solvable stochastic optimal stopping problem was presented and it was shown that the model is exactly solvable. Its possible extensions to more realistic population dynamics modeling were also discussed. There exist many other issues where the mathematical tools of financial research fields are effectively utilized, such as optimal usage of water resources (Unami et al., 2015; Sharifi et al., 2016), optimal management of fishery resources (Yaegashi et al., 2016;

Yaegashi et al., 2017), and optimal management of harmful bottom-attached algae in rivers (Yoshioka and Yaegashi, 2017a).

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